

# Angewandte Umweltsystemanalyse: Finite-Elemente-Methode (FEM) #3

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# Vorlesungsplan SoSe 2012: USA-I Numerische Methoden

#	Datum	Vorlesung	Übung	Skript
1	13.04.12	Einführung, Systemanalyse		
2	13.04.12	Grundwasserhydraulik, Prinzip-Beispiel	USA1	1.1+2
3	20.04.12	Einführung in geotechnische Modellierung		PDF
4	27.04.12	FDM#1: 2D, explizit	USA2	1.3
5	27.04.12	FDM#2: Selke-Modell, Q&D	USA3	1.4
6	04.05.12	FDM#3: Selke-Modell, OOP,	USA4	1.5
7	11.05.12	FDM#3: Selke-Modell, VTK	USA5	1.5
8	18.05.12	FH-DGG Tagung		
9	25.05.12	FDM#4: implizit, instationär	USA6	1.6
10	01.06.12	vorlesungsfrei		
11	08.06.12	FDM#5: stationär, Randbedingungen	USA7+8	1.7+8
12	08.06.12	FDM#6: Rechteck-Aquifer	USA9+10	1.9+10
13	15.06.12	FEM#1: Grundlagen		2.1
14	15.06.12	FEM#2: 1D Modell	USA11	2.2
15	22.06.12	FEM#3: Implementierung, 2D Modell	USA12	2.3
16	29.06.12	UFZ-Exkursion: MOSAIC / TERENO		
17	06.07.12	Qt, Klausurvorbereitung		
18	13.07.12	UFZ-Exkursion: VISLab		

# Lecture Table of Contents

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# Finite-Elemente-Methode (FEM)

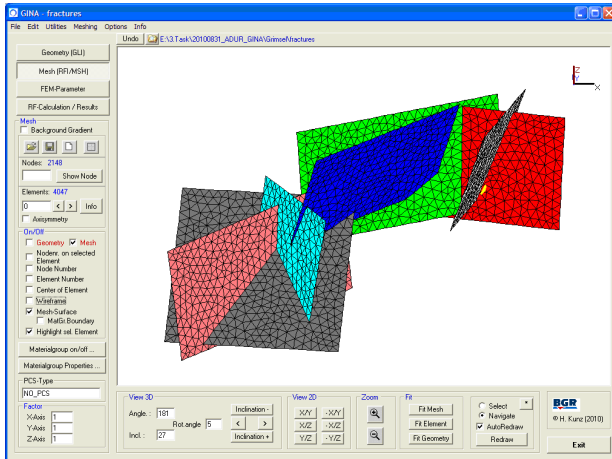


Abbildung: Modellierung eines Kluftsystems im Kristallin (Herbert Kunz, BGR)

# FEM: Element-Typen

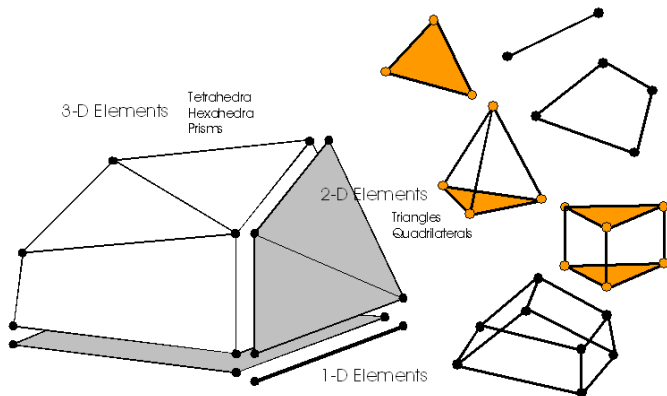


Abbildung: Mögliche Elementtypen für die Finite Elemente Methode (FEM)

# FEM: 2D Problem

Wir betrachten ein 2D stationäres Grundwasserströmungsproblem.

$$\frac{\partial}{\partial x} \left( K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( K_y \frac{\partial h}{\partial y} \right) = 0 \quad (1)$$

# FEM: Residuum - Schwache Formulierung

$$\int_{\Omega^e} N \left( \frac{\partial}{\partial x} \left( K_x \frac{\partial \hat{h}}{\partial x} \right) + \frac{\partial}{\partial y} \left( K_y \frac{\partial \hat{h}}{\partial y} \right) \right) d\Omega^e = R^e \approx 0 \quad (2)$$

$$\int_{\Omega^e} N(x, y) \left( K_x \frac{\partial^2 \hat{h}}{\partial x^2} + K_y \frac{\partial^2 \hat{h}}{\partial y^2} \right) d\Omega^e = R^e \approx 0 \quad (3)$$

K: Warum können wir das tun?

## FEM: Partielle Integration

Dabei nutzen wir zwei mathematische 'Tricks', die uns schon bekannt sind (Hydroinformatik II)

1. Kettenregel (integration by parts)

$$\begin{aligned}\nabla(\mathbf{N}A) &= \mathbf{N}\nabla A + \nabla\mathbf{N}A \\ \mathbf{N}\nabla A &= \underbrace{\nabla(\mathbf{N}A)}_B - \nabla\mathbf{N}A\end{aligned}\quad (4)$$

2. Definition der Divergenz

$$\int_{\Omega} \nabla B d\Omega = \oint_{\partial\Omega} B d\Gamma \quad (5)$$

3. Darcy's law

$$A = \mathbf{K}\nabla h \quad (6)$$

4. Gradient

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)^T \quad (7)$$



## FEM:

$$\int_{\Omega^e} N(x, y) \left( K_x \frac{\partial^2 \hat{h}}{\partial x^2} + K_y \frac{\partial^2 \hat{h}}{\partial y^2} \right) d\Omega^e = 0 \quad (8)$$

Die Gleichung lässt sich für 2D und 3D Problemstellungen erweitern

$$\int_{\Omega} \mathbf{N} \nabla (\mathbf{K} \nabla \mathbf{h}) d\Omega = - \int_{\Omega} \nabla \mathbf{N} \mathbf{K} \nabla \mathbf{h} d\Omega + \oint_{\partial\Omega} \mathbf{N} \mathbf{K} \nabla \mathbf{h} d\Gamma = 0 \quad (9)$$

# FEM: Näherungslösung

$$\int_{\Omega} \nabla \mathbf{N} \mathbf{K} \nabla \mathbf{h} d\Omega = \oint_{\partial\Omega} \mathbf{N} \mathbf{K} \nabla \mathbf{h} d\Gamma \quad (10)$$

$$h(x, y, t) = \sum_{i=1}^3 N_i(x, y) \hat{h}_i(t) \quad (11)$$

$$\mathbf{h}(x, y) = \mathbf{N}(x, y) \hat{\mathbf{h}}(t) \quad (12)$$

$$\int_{\Omega} (\nabla \mathbf{N} \mathbf{K} \nabla \mathbf{N}) \hat{\mathbf{h}} d\Omega = \oint_{\partial\Omega} (\mathbf{N} \mathbf{K} \nabla \mathbf{N}) \hat{\mathbf{h}} d\Gamma \quad (13)$$

## FEM: Symbols

$$\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}, \quad \hat{\mathbf{h}} = \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix}, \quad \mathbf{K} = \begin{Bmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{Bmatrix} \quad (14)$$

# FEM: Bildchens

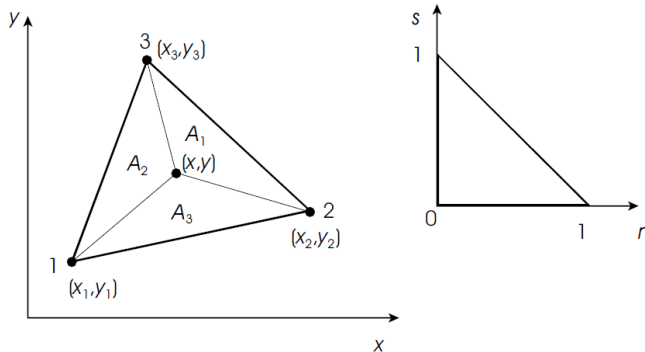


Abbildung: Linear triangular element

## FEM: Let's talk about **N**

Obviously, an arbitrary point in the triangle can be identified by use of the following local (area) coordinates (Fig. 3),

$$\begin{aligned}N_1 &= A_1/A \\N_2 &= A_2/A \\N_3 &= A_3/A\end{aligned}\tag{15}$$

where  $A$  is the area of triangular element

$$A = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\tag{16}$$

## FEM:

From geometrical reasons we have

$$1 = N_1 + N_2 + N_3 \quad (17)$$

Furthermore we can write

$$N_i = \begin{cases} 1 & : \text{ at node } i \\ 0 & : \text{ at remaining nodes} \end{cases} \quad (18)$$

- ▶ Bildchens

## FEM:

From this condition it can be concluded that

$$\begin{aligned}1 &= N_1 + N_2 + N_3 \\x &= N_1x_1 + N_2x_2 + N_3x_3 \\y &= N_1y_1 + N_2y_2 + N_3y_3\end{aligned}\tag{19}$$

Now we write the above equations in following compact matrix form

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}\tag{20}$$

## FEM:

Inversion gives

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \quad (21)$$

$$\mathbf{x} = \mathbf{AN}$$

$$\mathbf{N} = \mathbf{A}^{-1}\mathbf{x} \quad (22)$$



## FEM:

$$\begin{aligned}N_1(x, y) &= \frac{1}{2A} [(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \\N_2(x, y) &= \frac{1}{2A} [(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \\N_3(x, y) &= \frac{1}{2A} [(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] \quad (23)\end{aligned}$$

## FEM:

Now the derivatives of the shape functions can be easily written down.

$$\frac{\partial \mathbf{N}}{\partial x} = \left\{ \begin{array}{l} \frac{\partial N_1}{\partial x} = \frac{y_2 - y_3}{2A} \\ \frac{\partial N_2}{\partial x} = \frac{y_3 - y_1}{2A} \\ \frac{\partial N_3}{\partial x} = \frac{y_1 - y_2}{2A} \end{array} \right\} \quad \frac{\partial \mathbf{N}}{\partial y} = \left\{ \begin{array}{l} \frac{\partial N_1}{\partial y} = \frac{x_3 - x_2}{2A} \\ \frac{\partial N_2}{\partial y} = \frac{x_1 - x_3}{2A} \\ \frac{\partial N_3}{\partial y} = \frac{x_2 - x_1}{2A} \end{array} \right\} \quad (24)$$

# FEM: Stiffness Matrix #1

$$\mathbf{K}^e = \int_{\Omega^e} \nabla \mathbf{N} \mathbf{K} \nabla \mathbf{N}^T d\Omega^e \quad (25)$$

# FEM: Stiffness Matrix #2

$$\begin{aligned}
 \mathbf{K}^e &= \int_{\Omega^e} \nabla \mathbf{N} \mathbf{K} \nabla \mathbf{N}^T d\Omega^e & (26) \\
 &= \frac{K_{xx}^e}{4A} \begin{bmatrix} (y_2 - y_3)(y_2 - y_3) & (y_2 - y_3)(y_3 - y_1) & (y_2 - y_3)(y_1 - y_2) \\ (y_3 - y_1)(y_2 - y_3) & (y_3 - y_1)(y_3 - y_1) & (y_3 - y_1)(y_1 - y_2) \\ (y_1 - y_2)(y_2 - y_3) & (y_1 - y_2)(y_3 - y_1) & (y_1 - y_2)(y_1 - y_2) \end{bmatrix} \\
 &+ \frac{K_{yy}^e}{4A} \begin{bmatrix} (x_3 - x_2)(x_3 - x_2) & (x_3 - x_2)(x_1 - x_3) & (x_3 - x_2)(x_2 - x_1) \\ (x_1 - x_3)(x_3 - x_2) & (x_1 - x_3)(x_1 - x_3) & (x_1 - x_3)(x_2 - x_1) \\ (x_2 - x_1)(x_3 - x_2) & (x_2 - x_1)(x_1 - x_3) & (x_2 - x_1)(x_2 - x_1) \end{bmatrix}
 \end{aligned}$$

## FEM: Stiffness Matrix #3

$$\begin{aligned}
 K_{ij}^e &= \int_{\Omega^e} \frac{\partial N_i}{\partial x_\alpha} K_{\alpha\beta} \frac{\partial N_j}{\partial x_\beta} d\Omega^e & (27) \\
 &= \int_{\Omega^e} \frac{\partial N_i}{\partial x} (K_x \frac{\partial N_j}{\partial x} + K_y \frac{\partial N_j}{\partial y}) d\Omega^e + \int_{\Omega^e} \frac{\partial N_i}{\partial y} (K_x \frac{\partial N_j}{\partial x} + K_y \frac{\partial N_j}{\partial y}) d\Omega^e \\
 &= K_x \int_{\Omega^e} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} d\Omega^e + K_y \int_{\Omega^e} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} d\Omega^e \\
 &= K_x \int_{\Omega^e} \frac{y_j - y_k}{2A} \frac{y_k - y_i}{2A} d\Omega^e + K_y \int_{\Omega^e} \frac{x_k - x_j}{2A} \frac{x_i - x_k}{2A} d\Omega^e \\
 &= \frac{K_x}{4A} (y_j - y_k)(y_k - y_i) + \frac{K_y}{4A} (x_k - x_j)(x_i - x_k)
 \end{aligned}$$

# FEM: Exercise

tbd