Hydroinformatik II: Finite Differenzen Methode

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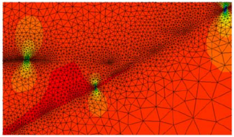
Vorlesungsplan Hydroinformatik II SoSe 2016

#	Datum	Thema
01	08.04.2016	Einführung
02	15.04.2016	Grundlagen: Kontinuumsmechanik
03	22.04.2016	Grundlagen: Hydromechanik
04	29.04.2016	Grundlagen: Partielle Differentialgleichungen / TEX
05	06.05.2016	HW: Qt Installation (2016)
06	13.05.2016	Qt Übung: Funktionsrechner; Grundlagen Numerik
07	20.05.2016	Pfingsten
08	27.05.2016	Numerik: (exp) Finite Differenzen Methode
09	03.06.2016	Vorbereitung für den Beleg
10	10.06.2016	Numerik: (imp) Finite Differenzen Methode
11	17.06.2016	Gerinnehydraulik: Theorie - Grundlagen
12	24.06.2016	Gerinnehydraulik: Programmierung, Übung 1
13	01.07.2016	Gerinnehydraulik: Programmierung, Übung 2
14	08.07.2016	Gerinnehydraulik: Programmierung, Übung 3
15	15.07.2016	Kurs-Zusammenfassung, Ausblick und Beleg

Konzept

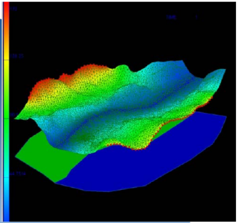
$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \mathbf{v}^E \nabla \psi$$



Basics

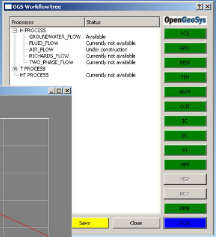
Mechanik

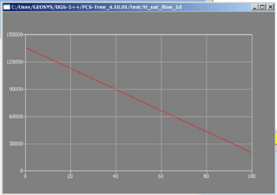
Anwendung



Numerische

Methoden





Programmierung

Visual C++

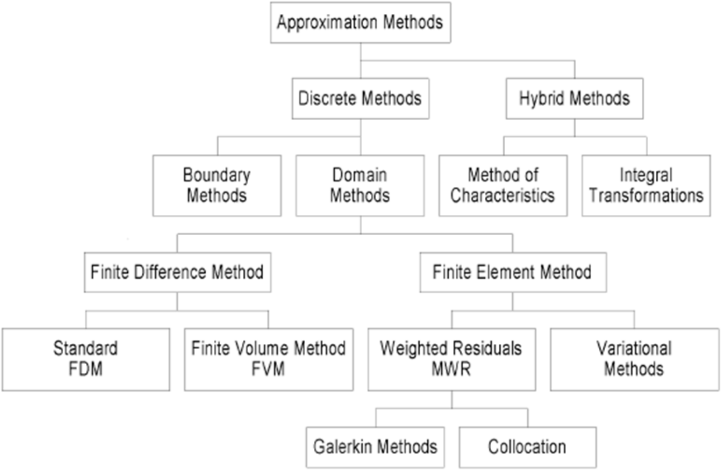
Prozessverständnis

Fahrplan

Vorlesung

- Grundlagen der Finite Differenzen Methode
- Approximation methods
- Finite difference method FDM (Ch. 3)
- Taylor series expansion
- Derivatives
- Diffusion equation
- (Finite element method FEM ⇒ Hydrosystemanalyse)

Näherungsverfahren



FDM Anwendungen - MODFLOW

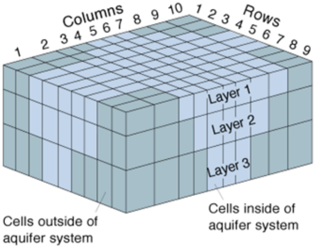


Figure 2. Example of model grid for simulating three-dimensional ground-water flow.

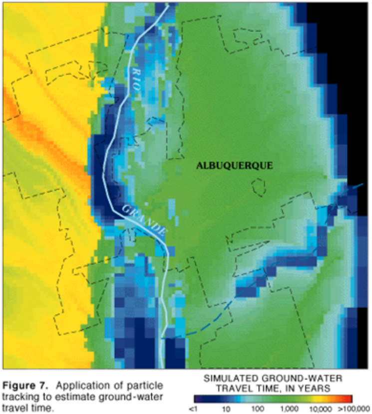
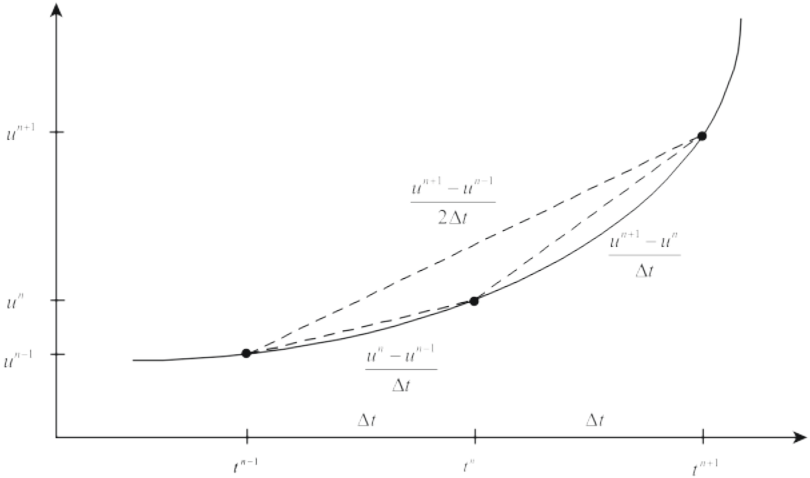


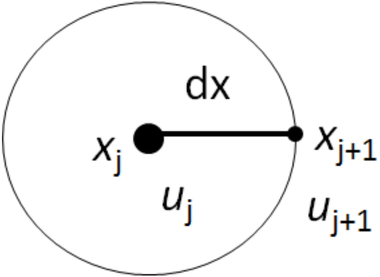
Figure 7. Application of particle tracking to estimate ground-water travel time.

http://water.usgs.gov/pubs/FS/FS-121-97/images/fig7.gif

Ableitungen



Taylor-Reihe



in time

$$u_j^{n+1} = \sum_{m=0}^{\infty} \frac{\Delta t^m}{m!} \left[\frac{\partial^m u}{\partial t^m} \right]_j^n \quad (1)$$

in space

$$u_{j+1}^{n} = \sum_{m=0}^{\infty} \frac{\Delta x^m}{m!} \left[\frac{\partial^m u}{\partial x^m} \right]_j^n \quad (2)$$

Trunkation

$$u_j^{n+1} = u_j^n + \Delta t \left[\frac{\partial u}{\partial t}\right]_j^n + \frac{\Delta t^2}{2} \left[\frac{\partial^2 u}{\partial t^2}\right]_j^n + 0(\Delta t^3) \tag{3}$$

$$u_{j+1}^n = u_j^n + \Delta x \left[\frac{\partial u}{\partial x}\right]_j^n + \frac{\Delta x^2}{2} \left[\frac{\partial^2 u}{\partial x^2}\right]_j^n + 0(\Delta x^3) \tag{4}$$

1. Ableitung

$$\left[\frac{\partial u}{\partial t}\right]_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\Delta t}{2}\left[\frac{\partial^2 u}{\partial t^2}\right]_j^n + 0(\Delta t^2) \tag{5}$$

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} - \frac{\Delta x}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n + 0(\Delta x^2) \tag{6}$$

Differenzen-Schemata

Forward difference approximation

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} + 0(\Delta x) \tag{7}$$

Backward difference approximation

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x} + 0(\Delta x) \tag{8}$$

Central difference approximation

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + 0(\Delta x^2) \tag{9}$$

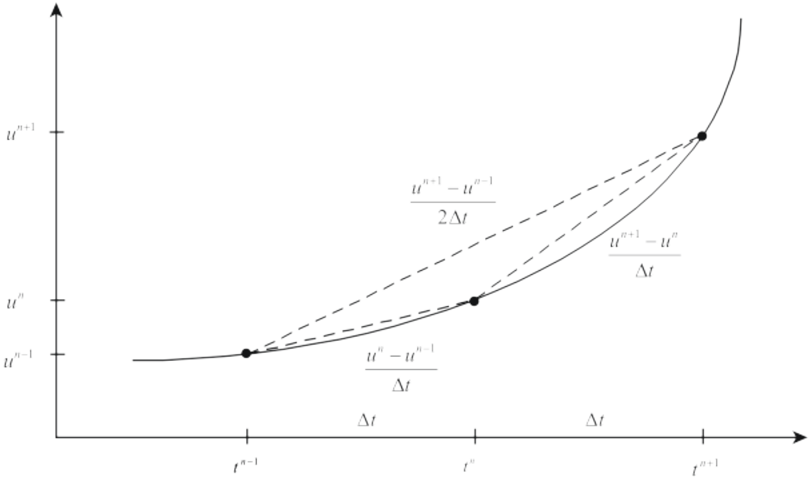
Zentrale Differenzen

$$
\begin{aligned}
u_{j+1}^n &= u_j^n + \Delta x \left[\frac{\partial u}{\partial x}\right]_j^n + \frac{\Delta x^2}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n + 0(\Delta x^3) \\
u_{j-1}^n &= u_j^n - \Delta x \left[\frac{\partial u}{\partial x}\right]_j^n + \frac{\Delta x^2}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n - 0(\Delta x^3)
\end{aligned} \tag{10}
$$

Central difference approximation

$$
\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + 0(\Delta x^2) \tag{11}
$$

Ableitungen

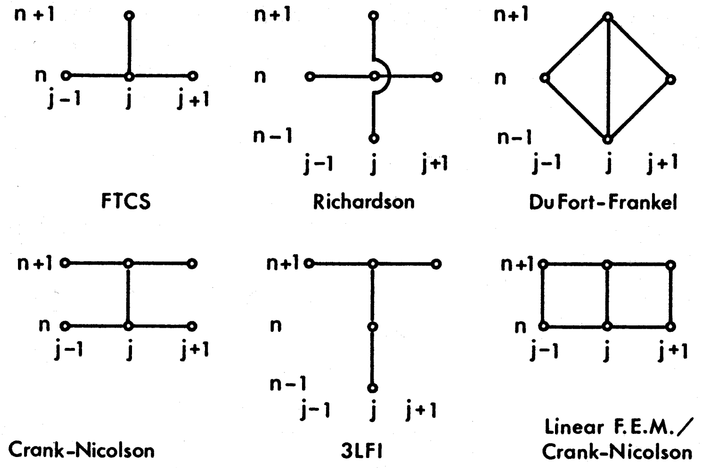


2. Ableitung

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n \approx \frac{1}{\Delta x}\left(\left[\frac{\partial u}{\partial x}\right]_{j+1}^n - \left[\frac{\partial u}{\partial x}\right]_j^n\right) \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \qquad (12)$$

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{\Delta x^2}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^n + \dots \qquad (13)$$

Übersicht Differenzenverfahren



Diffusionsgleichung

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \qquad (14)$$

Analytical solution for diffusion equation (Skript 5.2.2)

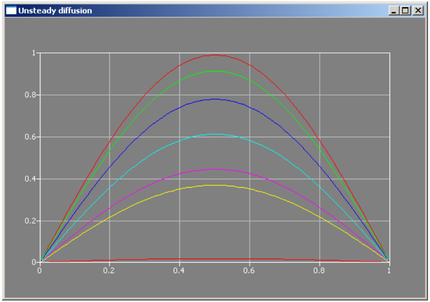
- Diffusion equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \qquad (15)$$

- Analytical solution

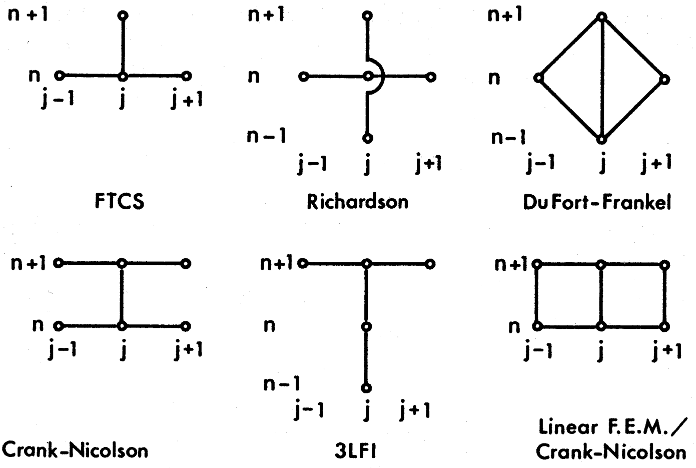
$$u = sin(\pi x) e^{-\alpha t^2} \qquad (16)$$

- K: validity



⇒ Übung

Übersicht Differenzenverfahren



Explizite FDM - FTCS Verfahren (Skript 3.2.2/4.1)

- PDE for diffusion processes

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \tag{17}$$

- forward time / centered space

$$\left[\frac{\partial u}{\partial t}\right]_j^n \approx \frac{u_j^{n+1} - u_j^n}{\Delta t} \qquad \left[\frac{\partial^2 u}{\partial x^2}\right]_j^n \approx \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} \tag{18}$$

- substitute

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} = 0 \tag{19}$$

- FTCS scheme for diffusion equations

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n) \qquad \boxed{Ne = \frac{\alpha \Delta t}{\Delta x^2}} \tag{20}$$

Eigenschaften numerischer Verfahren

Analysis of approximation schemes consists of three steps:

- Develop the **algebraic scheme**,
- Check **consistency** of the algebraic approximate equation,
- Investigate **stability** behavior of the scheme.

Eigenschaften numerischer Verfahren

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$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n) \tag{21}$$

- Check **consistency** of the algebraic approximate equation,

$$\lim_{\Delta t, \Delta x \to 0} |\hat{L}(u_j^n) - L(u[t_n, x_j])| = 0 \tag{22}$$

- Investigate **stability** behavior of the scheme.

$$\boxed{Ne = \frac{\alpha \Delta t}{\Delta x^2} \leq 1/2} \tag{23}$$

Lösung des FTCS Schemas

Algebraische Schema

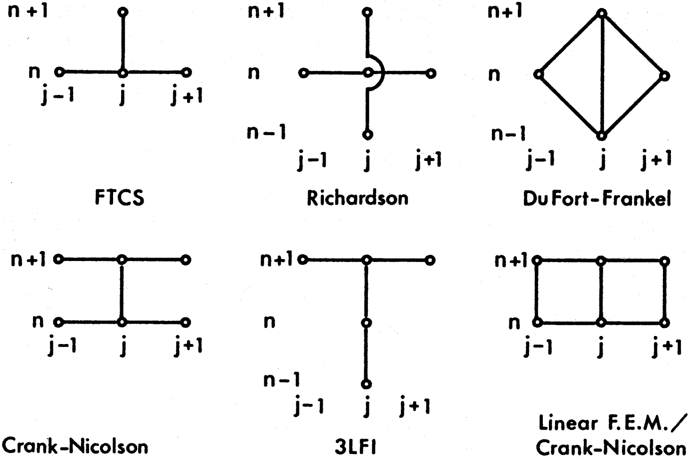
$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n) \tag{24}$$

Resultierendes Gleichungssystem

$$\mathbf{u}^{n+1} = \mathbf{A}\mathbf{u}^n \quad , \quad n = 0, 1, 2, \ldots \tag{25}$$

$$\mathbf{A} = \begin{bmatrix} 1 - 2\frac{\alpha \Delta t}{\Delta x^2} & \frac{\alpha \Delta t}{\Delta x^2} & & & \\ \frac{\alpha \Delta t}{\Delta x^2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{\alpha \Delta t}{\Delta x^2} \\ & & & \frac{\alpha \Delta t}{\Delta x^2} & 1 - 2\frac{\alpha \Delta t}{\Delta x^2} \end{bmatrix}, \; \mathbf{u}^n = \begin{bmatrix} u_2^n \\ u_3^n \\ \ldots \\ u_{np-2}^n \\ u_{np-1}^n \end{bmatrix}$$

Explizite und implizite Differenzenverfahren



Implizites Differenzenverfahren: Next Lecture

Algebraische Schema:

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n+1} \approx \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} \tag{26}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} = 0 \tag{27}$$

$$\frac{\alpha \Delta t}{\Delta x^2} \left(-u_{j-1}^{n+1} + 2u_j^{n+1} - u_{j+1}^{n+1}\right) + u_j^{n+1} = u_j^n \tag{28}$$

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$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} = 0 \quad (27)$$

$$\frac{\alpha \Delta t}{\Delta x^2}\left(-u_{j-1}^{n+1} + 2u_j^{n+1} - u_{j+1}^{n+1}\right) + u_j^{n+1} = u_j^n \quad (28)$$

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Algebraische Schema:

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n+1} \approx \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} \tag{26}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} = 0 \tag{27}$$

$$\frac{\alpha \Delta t}{\Delta x^2} \left(-u_{j-1}^{n+1} + 2u_j^{n+1} - u_{j+1}^{n+1}\right) + u_j^{n+1} = u_j^n \tag{28}$$