BHYWI-08: Semester-Fahrplan

Vorlesungen

Datum	V	Thema
13.04.2018	01	Einführung / Qt Installation
20.04.2018	02	Grundlagen: Kontinuumsmechanik
27.04.2018	03	Grundlagen: Hydromechanik
04.05.2018	04	Grundlagen: Partielle Differentialgleichungen
11.05.2018	05	Grundlagen: Numerik, Qt Übung: Funktionsrechner
18.05.2018	06	Numerik: Finite Differenzen Methode I (explizit)
01.06.2018	07	Numerik: Finite Differenzen Methode II (implizit)
08.06.2018	08	Gerinnehydraulik: Theorie – Grundlagen
15.06.2018	09	Gerinnehydraulik: Programmierung, Übung 1
22.06.2018	10	Gerinnehydraulik: Programmierung, Übung 2
29.06.2018	11	Grundwassermodellierung: Catchment Übung
06.07.2018	12	Grundwassermodellierung: Datenbasierte Methoden I
13.07.2018	13	Grundwassermodellierung: Datenbasierte Methoden II
20.07.2018	14	Klausurvorbereitung

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Hydroinformatik II

"Prozesssimulation und Systemanalyse"

BHYWI-08-06 @ 2018

Finite-Differenzen-Verfahren I

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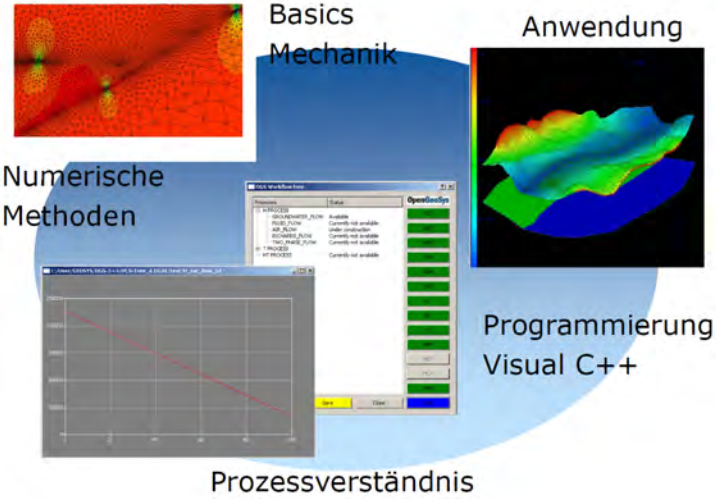
[2]Centre for Advanced Water Research – CAWR

18.05./01.06.2018 - Dresden

Konzept

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}^E \nabla\psi$$



Fahrplan

Vorlesung

- Grundlagen der Finite Differenzen Methode
- Approximation methods
- Finite difference method – FDM (Ch. 3)
- Taylor series expansion
- Derivatives
- Diffusion equation
- (Finite element method – FEM ⇒ Hydrosystemanalyse)

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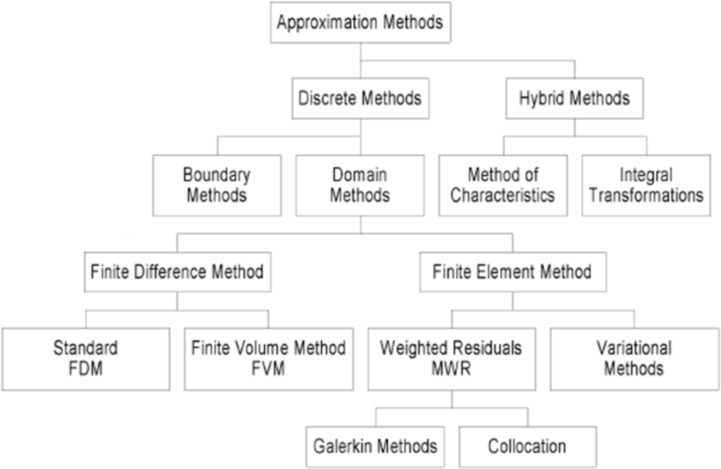
Näherungsverfahren



FDM Anwendungen - MODFLOW

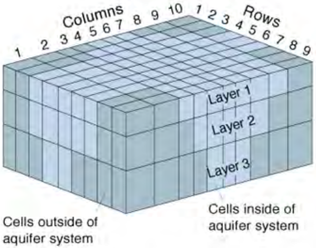


Figure 2. Example of model grid for simulating three-dimensional ground-water flow.

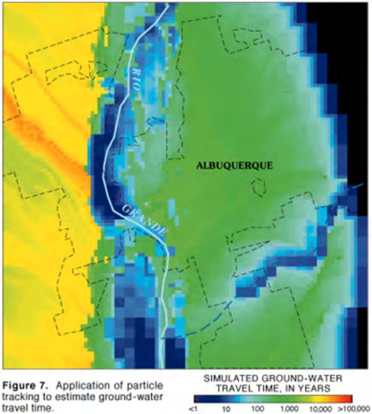


Figure 7. Application of particle tracking to estimate ground-water travel time.

http://water.usgs.gov/pubs/FS/FS-121-97/images/fig7.gif

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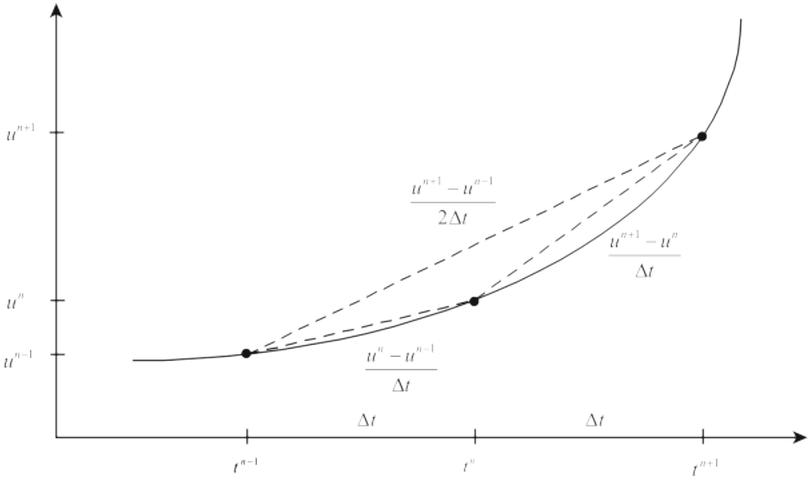
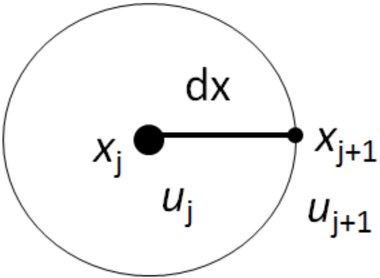
Ableitungen



Taylor-Reihe



in time

$$u_j^{n+1} = \sum_{m=0}^{\infty} \frac{\Delta t^m}{m!} \left[\frac{\partial^m u}{\partial t^m}\right]_j^n \qquad (1)$$

$$\Delta t = t^{n+1} - t^n$$

in space

$$u_{j+1}^{n} = \sum_{m=0}^{\infty} \frac{\Delta x^m}{m!} \left[\frac{\partial^m u}{\partial x^m}\right]_j^n \qquad (2)$$

$$\Delta x = x_{j+1} - x_j$$

Trunkation

$$u_j^{n+1} = u_j^n + \Delta t \left[\frac{\partial u}{\partial t}\right]_j^n + \frac{\Delta t^2}{2}\left[\frac{\partial^2 u}{\partial t^2}\right]_j^n + 0(\Delta t^3) \qquad (3)$$

$$u_{j+1}^{n} = u_j^n + \Delta x \left[\frac{\partial u}{\partial x}\right]_j^n + \frac{\Delta x^2}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n + 0(\Delta x^3) \qquad (4)$$

1. Ableitung

$$\left[\frac{\partial u}{\partial t}\right]_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\Delta t}{2}\left[\frac{\partial^2 u}{\partial t^2}\right]_j^n + 0(\Delta t^2) \quad (5)$$

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^{n} - u_j^n}{\Delta x} - \frac{\Delta x}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n + 0(\Delta x^2) \quad (6)$$

Forward difference approximation

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} + 0(\Delta x) \tag{7}$$

Backward difference approximation

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x} + 0(\Delta x) \tag{8}$$

Central difference approximation

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + 0(\Delta x^2) \tag{9}$$

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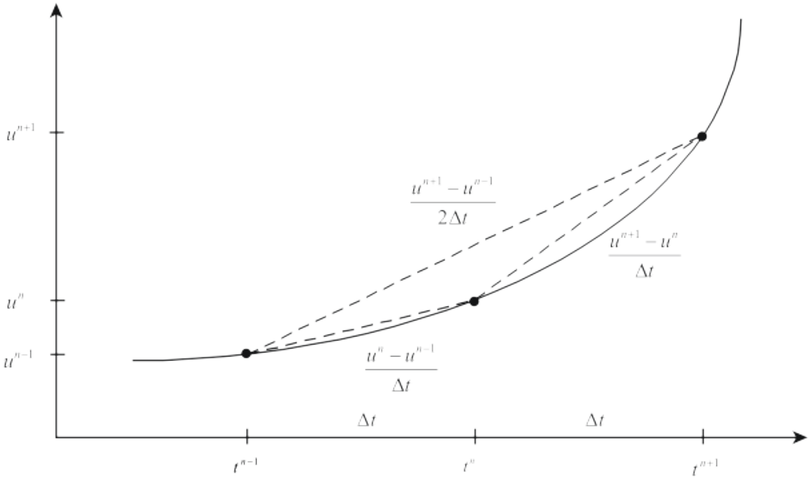
Zentrale Differenzen

$$
\begin{aligned}
u_{j+1}^n &= u_j^n + \Delta x \left[\frac{\partial u}{\partial x}\right]_j^n + \frac{\Delta x^2}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n + 0(\Delta x^3) \\
u_{j-1}^n &= u_j^n - \Delta x \left[\frac{\partial u}{\partial x}\right]_j^n + \frac{\Delta x^2}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n - 0(\Delta x^3)
\end{aligned} \tag{10}
$$

Central difference approximation

$$
\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + 0(\Delta x^2) \tag{11}
$$

Ableitungen

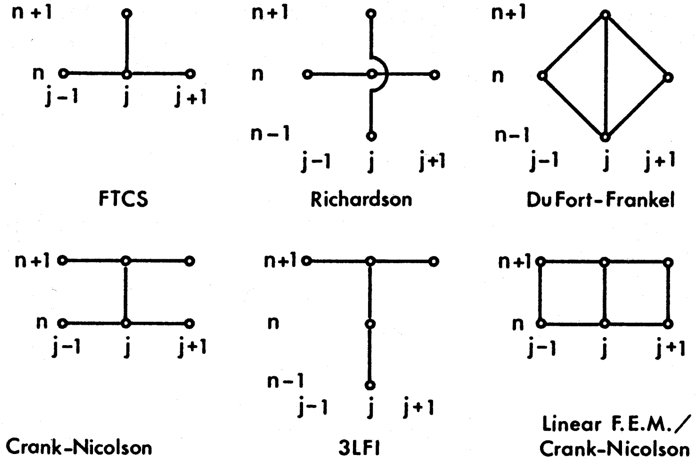


2. Ableitung

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n \approx \frac{1}{\Delta x}\left(\left[\frac{\partial u}{\partial x}\right]_{j+1}^n - \left[\frac{\partial u}{\partial x}\right]_j^n\right) \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \tag{12}$$

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{\Delta x^2}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^n + \dots \tag{13}$$

Übersicht Differenzenverfahren



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Diffusionsgleichung

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \tag{14}$$

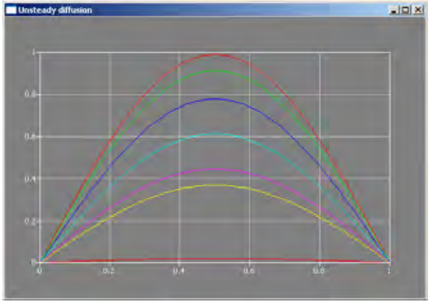
- Diffusion equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \qquad (15)$$

- Analytical solution

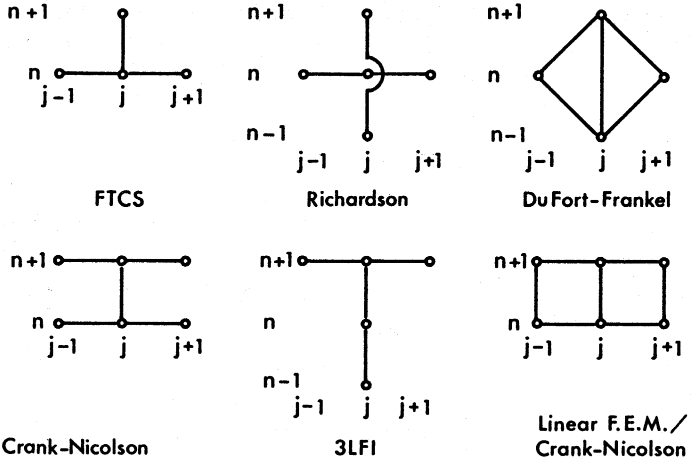
$$u = sin(\pi x) e^{-\alpha t^2} \qquad (16)$$

- K: validity



$\Rightarrow$ Übung

Übersicht Differenzenverfahren



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Explizite FDM - FTCS Verfahren (Skript 3.2.2/4.1)

- PDE for diffusion processes

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \tag{17}$$

- forward time / centered space

$$\left[\frac{\partial u}{\partial t}\right]_j^n \approx \frac{u_j^{n+1} - u_j^n}{\Delta t} \qquad \left[\frac{\partial^2 u}{\partial x^2}\right]_j^n \approx \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} \tag{18}$$

- substitute

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} = 0 \tag{19}$$

- FTCS scheme for diffusion equations

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right), \; Ne = \frac{\alpha \Delta t}{\Delta x^2} \tag{20}$$

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Analysis of approximation schemes consists of three steps:

- Develop the **algebraic scheme**,
- Check **consistency** of the algebraic approximate equation,
- Investigate **stability** behavior of the scheme.

Analysis of approximation schemes consists of three steps:

- Develop the **algebraic scheme**,

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n) \tag{21}$$

- Check **consistency** of the algebraic approximate equation,

$$\lim_{\Delta t, \Delta x \to 0} |\hat{L}(u_j^n) - L(u[t_n, x_j])| = 0 \tag{22}$$

- Investigate **stability** behavior of the scheme.

$$\boxed{\mathrm{Ne} = \alpha \Delta t \frac{}{\Delta x^2 \leq 1/2}} \tag{23}$$

Lösung des FTCS Schemas

Algebraische Schema

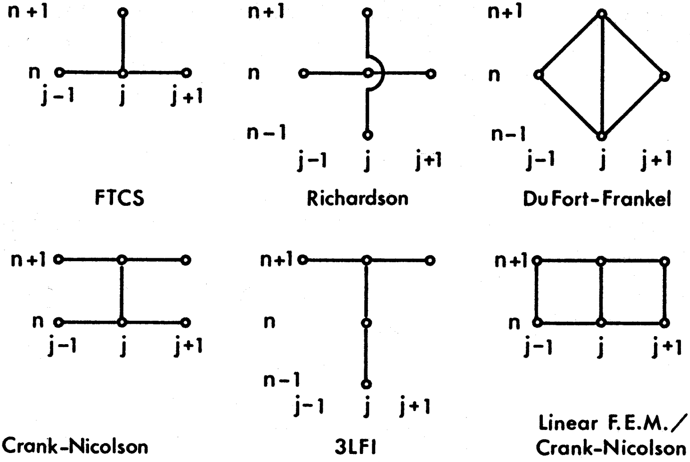
$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n) \tag{24}$$

Resultierendes Gleichungssystem

$$\mathbf{u}^{n+1} = \mathbf{A}\mathbf{u}^n \qquad , \qquad n = 0, 1, 2, ... \tag{25}$$

$$\mathbf{A} = \begin{bmatrix} 1-2 & & & & \\ & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & & 1-2 \end{bmatrix}, \mathbf{u}^n = \begin{bmatrix} u_2^n \\ u_3^n \\ \dots \\ u_{np-2}^n \\ u_{np-1}^n \end{bmatrix}$$

Explizite und implizite Differenzenverfahren



Algebraische Schema:

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n+1} \approx \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} \tag{26}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} = 0 \tag{27}$$

$$\frac{\alpha \Delta t}{\Delta x^2}\left(-u_{j-1}^{n+1} + 2u_j^{n+1} - u_{j+1}^{n+1}\right) + u_j^{n+1} = u_j^n \tag{28}$$

BHYWI-08: Semester-Fahrplan

Vorlesungen

Datum	E	Übungen
11.05.2018	01	Qt: Hallo World
11.05.2018	02	Qt: Funktionsrechner
18.05.2018	03	Qt: Explizite Finite-Differenzen-Methode
01.06.2018	04	Qt: Implizite Finite-Differenzen-Methode
15.06.2018	05	Qt: Gerinnehydraulik I (QAD)
22.06.2018	06	Qt: Gerinnehydraulik II (OOP)
22.06.2018	07	Qt: Gerinnehydraulik III (interaktiv)
29.06.2018	08	Qt: Gerinnehydraulik IV (interaktiv)

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