Hydroinformatik II

"Prozesssimulation und Systemanalyse"

BHYWI-08-08 @ 2020

Finite-Differenzen-Methode

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12.06.2020 - Dresden

Zeitplan: Hydroinformatik II

Datum	V	Thema	.
17.04.2020	00	Einführung in GoToMeeting (Web-Conferencing)	
17.04.2020	01	Einführung in die Lehrveranstaltung	
24.04.2020	02	Grundlagen: Kontinuumsmechanik	
08.05.2020	03	Grundlagen: Hydromechanik	
15.05.2020	04	Grundlagen: Partielle Partialgleichungen	
22.05.2020	05	Installation: Python, Qt C++	
29.05.2020	06	Grundlagen: Näherungsverfahren	
05.06.2020	07	Übungen: Übersicht und Werkzeuge	
12.06.2020	08	Numerik: Finite-Differenzen-Methode (explizit)	
19.06.2020	09	Numerik: Finite-Differenzen-Methode (implizit)	
26.06.2020	10	Anwendung: Gerinnehydraulik (Theorie)	
03.07.2020	11	Anwendung: Gerinnehydraulik (Übung)	
10.07.2020	12	Anwendung: Grundwassermodellierung (datenbasierte Methoden)	
17.07.2020	13	Beleg: Besprechung zur Vorbereitung	

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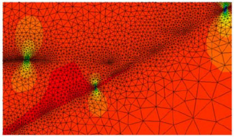
Konzept

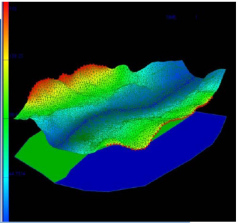
$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \mathbf{v}^E \nabla \psi$$



Basics

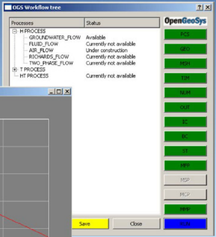
Mechanik

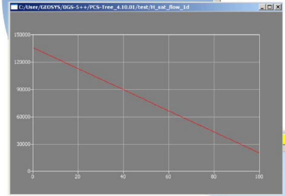
Anwendung



Numerische

Methoden





Programmierung

Visual C++

Prozessverständnis

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Fahrplan

Übung

- Python ...

Vorlesung

- Grundlagen der Finite Differenzen Methode
- Approximation methods
- Finite difference method – FDM (Ch. 3)
- Taylor series expansion
- Derivatives
- Diffusion equation
- (Finite element method – FEM $\Rightarrow$ Hydrosystemanalyse)

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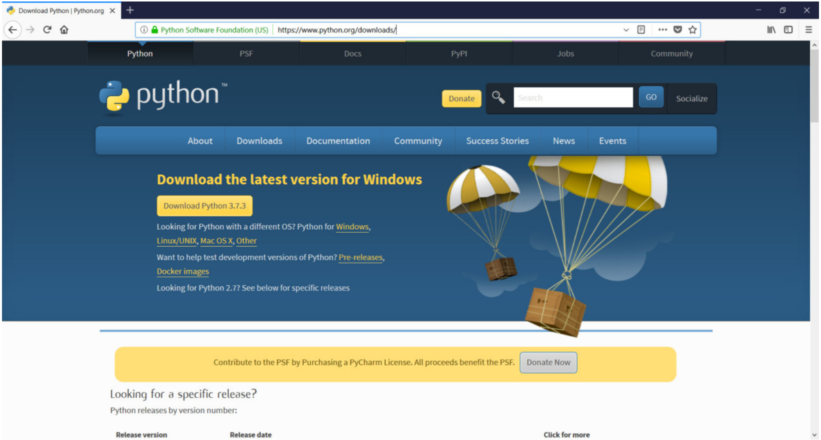
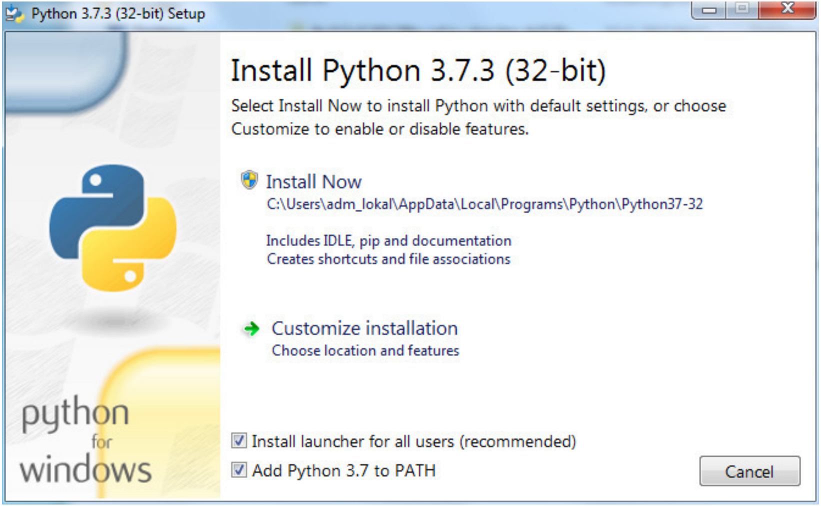
Python, die Zweite (Path)

https://www.python.org/downloads/



Python, die Zweite (Path)



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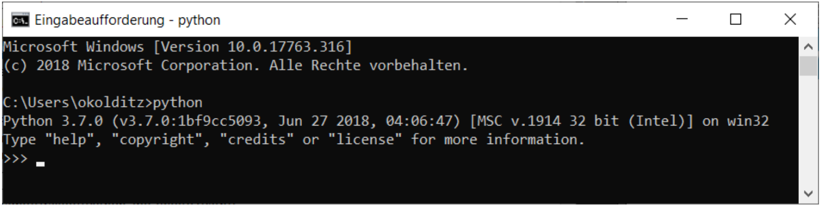
Python, die Zweite (Path)

Zwei Optionen:

1. Python deinstallieren, neu installieren und "Add Python to PATH"
2. "PATH" nachträglich ergänzen (unterschiedlich für verschiedene Windows-Versionen), am besten googeln



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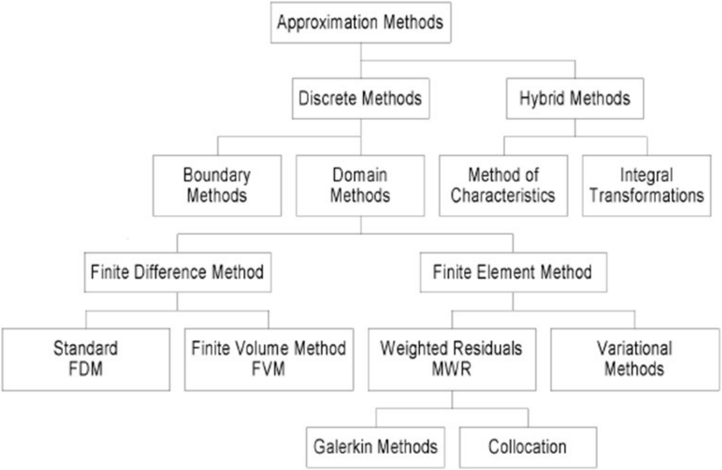
Näherungsverfahren



FDM Anwendungen - MODFLOW

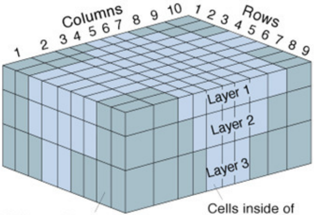


Figure 2. Example of model grid for simulating three-dimensional ground-water flow.

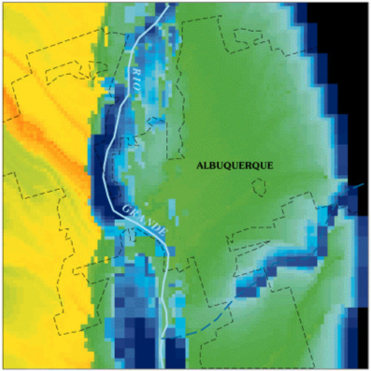


Figure 7. Application of particle tracking to estimate ground-water travel time.

http://water.usgs.gov/pubs/FS/FS-121-97/images/fig7.gif

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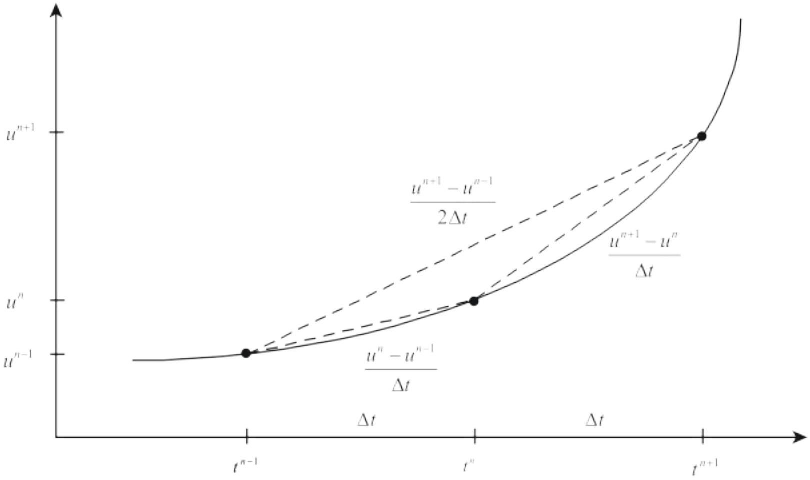
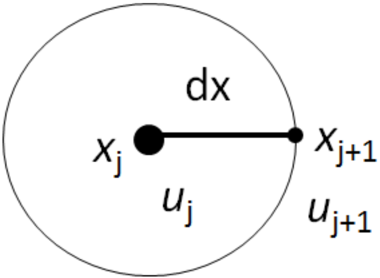
Ableitungen





in time

$$u_j^{n+1} = \sum_{m=0}^{\infty} \frac{\Delta t^m}{m!} \left[\frac{\partial^m u}{\partial t^m}\right]_j^n \quad (1)$$

$$\Delta t = t^{n+1} - t^n$$

in space

$$u_{j+1}^{n} = \sum_{m=0}^{\infty} \frac{\Delta x^m}{m!} \left[\frac{\partial^m u}{\partial x^m}\right]_j^n \quad (2)$$

$$\Delta x = x_{j+1} - x_j$$

Trunkation

$$u_j^{n+1} = u_j^n + \Delta t \left[\frac{\partial u}{\partial t}\right]_j^n + \frac{\Delta t^2}{2}\left[\frac{\partial^2 u}{\partial t^2}\right]_j^n + 0(\Delta t^3) \qquad (3)$$

$$u_{j+1}^n = u_j^n + \Delta x \left[\frac{\partial u}{\partial x}\right]_j^n + \frac{\Delta x^2}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n + 0(\Delta x^3) \qquad (4)$$

1. Ableitung

$$\left[\frac{\partial u}{\partial t}\right]_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\Delta t}{2}\left[\frac{\partial^2 u}{\partial t^2}\right]_j^n + 0(\Delta t^2) \qquad (5)$$

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^{n} - u_j^n}{\Delta x} - \frac{\Delta x}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n + 0(\Delta x^2) \qquad (6)$$

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Forward difference approximation

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x} + 0(\Delta x) \tag{7}$$

Backward difference approximation

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x} + 0(\Delta x) \tag{8}$$

Central difference approximation

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + 0(\Delta x^2) \tag{9}$$

Zentrale Differenzen

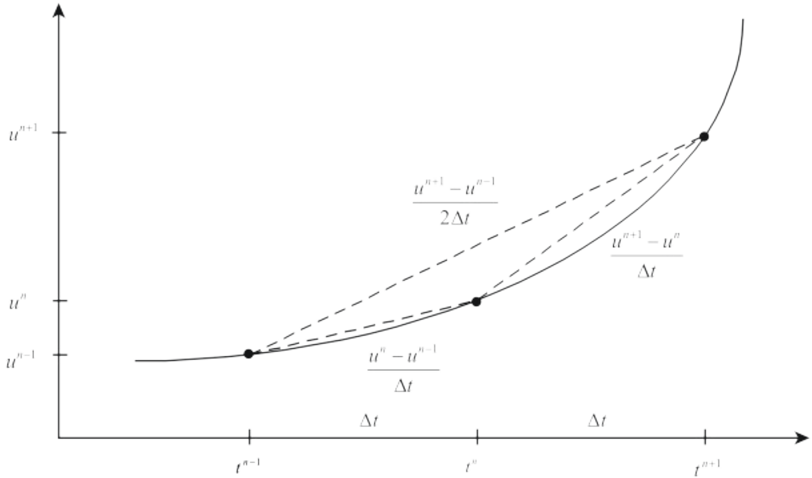
$$u_{j+1}^n = u_j^n + \Delta x \left[\frac{\partial u}{\partial x}\right]_j^n + \frac{\Delta x^2}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n + 0(\Delta x^3)$$

$$u_{j-1}^n = u_j^n - \Delta x \left[\frac{\partial u}{\partial x}\right]_j^n + \frac{\Delta x^2}{2}\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n - 0(\Delta x^3) \quad (10)$$

Central difference approximation

$$\left[\frac{\partial u}{\partial x}\right]_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + 0(\Delta x^2) \quad (11)$$

Ableitungen



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2. Ableitung

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n \approx \frac{1}{\Delta x}\left(\left[\frac{\partial u}{\partial x}\right]_{j+1}^n - \left[\frac{\partial u}{\partial x}\right]_j^n\right) \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \quad (12)$$

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{\Delta x^2}{12}\left[\frac{\partial^4 u}{\partial x^4}\right]_j^n + \dots \quad (13)$$

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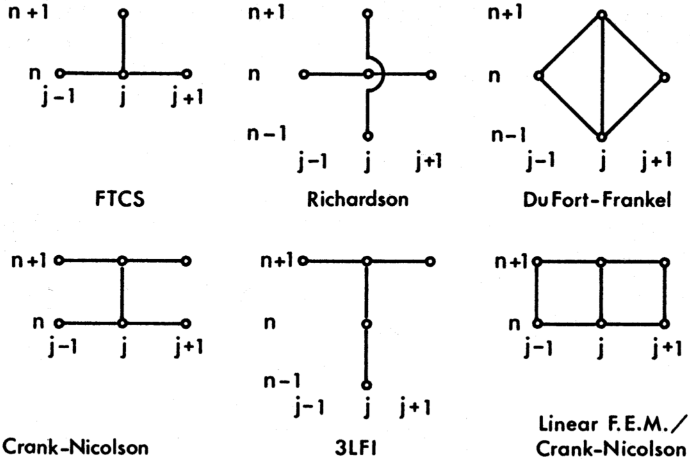
Übersicht Differenzenverfahren



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Diffusionsgleichung

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \qquad (14)$$

Analytical solution for diffusion equation (Skript 5.2.2)

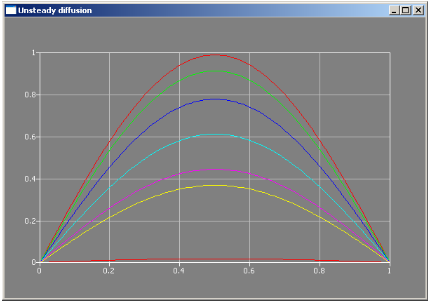
- Diffusion equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \qquad (15)$$

- Analytical solution

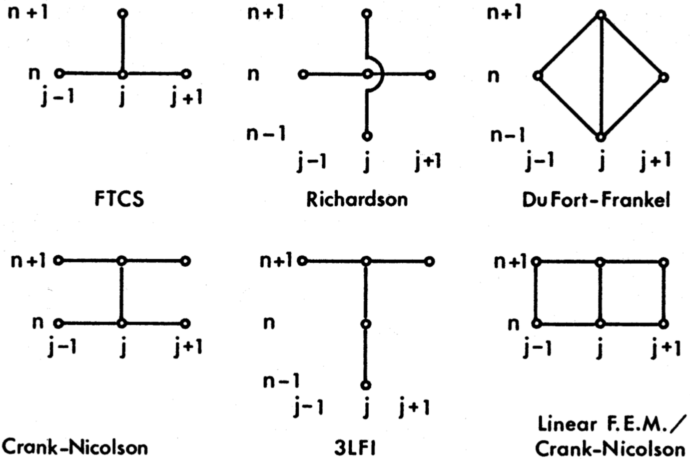
$$u = sin(\pi x)e^{-\alpha t^2} \qquad (16)$$

- K: validity



$\Rightarrow$ Übung

Übersicht Differenzenverfahren



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Explizite FDM - FTCS Verfahren (Skript 3.2.2/4.1)

- PDE for diffusion processes

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \qquad (17)$$

- forward time / centered space

$$\left[\frac{\partial u}{\partial t}\right]_j^n \approx \frac{u_j^{n+1} - u_j^n}{\Delta t} \quad \left[\frac{\partial^2 u}{\partial x^2}\right]_j^n \approx \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} \qquad (18)$$

- substitute

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} = 0 \qquad (19)$$

- FTCS scheme for diffusion equations

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right), \; Ne = \frac{\alpha \Delta t}{\Delta x^2} \qquad (20)$$

Eigenschaften numerischer Verfahren

Analysis of approximation schemes consists of three steps:

- Develop the **algebraic scheme**,
- Check **consistency** of the algebraic approximate equation,
- Investigate **stability** behavior of the scheme.

Analysis of approximation schemes consists of three steps:

- Develop the **algebraic scheme**,

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n) \tag{21}$$

- Check **consistency** of the algebraic approximate equation,

$$\lim_{\Delta t, \Delta x \to 0} |\hat{L}(u_j^n) - L(u[t_n, x_j])| = 0 \tag{22}$$

- Investigate **stability** behavior of the scheme.

$$Ne = \frac{\alpha \Delta t}{\Delta x^2} \leq 1/2 \tag{23}$$

Lösung des FTCS Schemas

Algebraische Schema

$$u_j^{n+1} = u_j^n + \frac{\alpha \Delta t}{\Delta x^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n) \tag{24}$$

Resultierendes Gleichungssystem

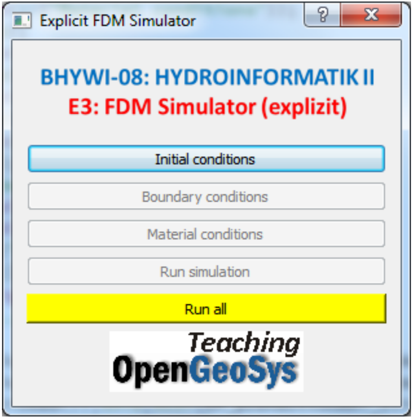
$$\mathbf{u}^{n+1} = \mathbf{A}\mathbf{u}^n \qquad , \qquad n = 0, 1, 2, ... \tag{25}$$

$$\mathbf{A} = \begin{bmatrix} 1 - 2\frac{\alpha\Delta t}{\Delta x^2} & \frac{\alpha\Delta t}{\Delta x^2} & & & \\ \frac{\alpha\Delta t}{\Delta x^2} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{\alpha\Delta t}{\Delta x^2} \\ & & & \frac{\alpha\Delta t}{\Delta x^2} & 1 - 2\frac{\alpha\Delta t}{\Delta x^2} \end{bmatrix}, \mathbf{u}^n = \begin{bmatrix} u_2^n \\ u_3^n \\ ... \\ u_{np-2}^n \\ u_{np-1}^n \end{bmatrix}$$

Übung BHYWI-08-03-E

FDM: Explizit

Progamm-Dialog



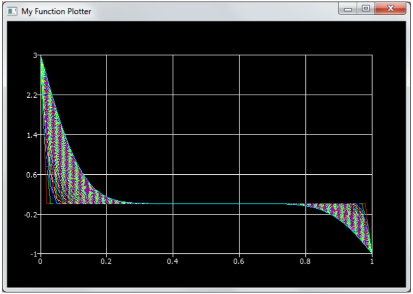
Dialog-Klasse: Konstruktor

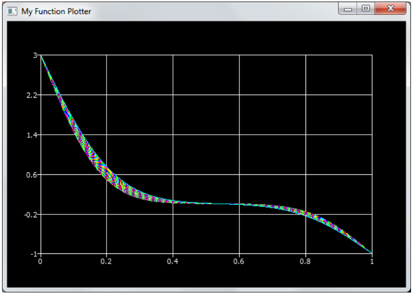
`Dialog::Dialog`

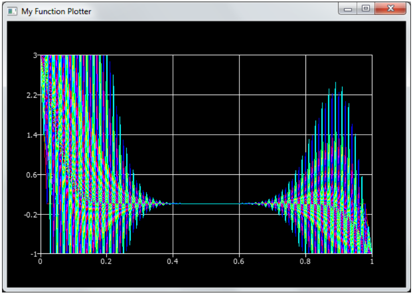
1. Elemente
2. Connects
3. Layout
4. Datenstrukturen (Speicherreservierung)

FDM: Explizit

Ergebnisse







$$Ne = \frac{\alpha \Delta t}{\Delta x^2} \leq 0.5 \qquad (26)$$

How sensitive ?

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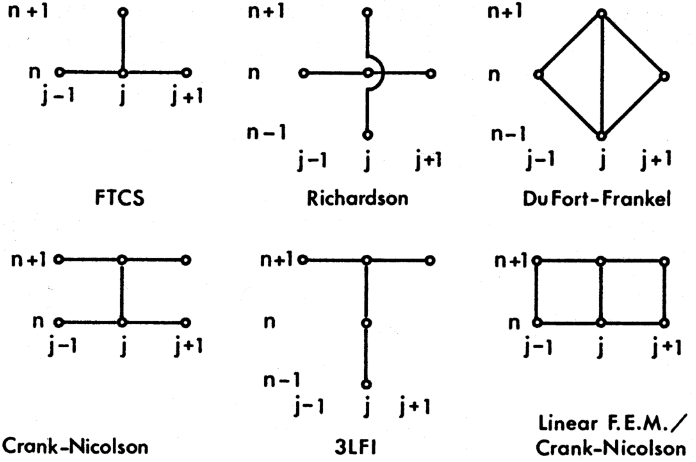
Explizite und implizite Differenzenverfahren



Implizites Differenzenverfahren: Next Lecture

Algebraische Schema:

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_j^{n+1} \approx \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} \tag{27}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \alpha \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{\Delta x^2} = 0 \tag{28}$$

$$\frac{\alpha \Delta t}{\Delta x^2} \left(-u_{j-1}^{n+1} + 2u_j^{n+1} - u_{j+1}^{n+1}\right) + u_j^{n+1} = u_j^n \tag{29}$$

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